Introduction to Quantum Intelligence

Michail Zak

Center for Space Microelectronics Technology
Jet Propulsion Laboratory
California Institute of Technology
Pasadena, CA 91109

Abstract

An impact of ideas associated with the concept of a hypothetical quantum computer upon classical computing is analyzed.

Two fundamental properties of quantum computing: direct simulations of probabilities, and influence between different branches of probabilistic scenarios, as well as their classical versions, are discussed. It has been demonstrated that classical probabilities, and in particular, probabilistic Turing machine, can be simulated by combining chaos and non-Lipschitz dynamics, without utilization of any man-made devices. Special attention was focused upon coupled stochastic processes defined in terms of conditional probabilities for which joint distributions do not exist. The effect of interference of probabilities is imitated by a special type of coupling between simulated and calculated probabilities. It is demonstrated that this classical imitation of a quantum effect can be useful in treatment of non-quantum problems of exponential complexity which occur in intelligent systems. Based upon this effect, the concept of quantum intelligence is introduced. New possible way of simulation of the Schrödinger equation is discussed.

1. Introduction

During the last fifty years, a theory of computations has been based upon classical physics implemented by the deterministic Turing machine. However, along with many successes of digital computers, the existence of so called hard problems put some limitations on their capabilities, since the computational time for such problems grows exponentially as a function of the dimensionality. It was well understood that the only way to fight the “curse” of the combinatorial explosion is to enrich digital computers with analog devices. In

contradistinction to a digital computer which performs operations on numbers symbolizing an underlying physical process, an analog computer processes information by exploiting physical phenomena directly, and thereby, it significantly reduces the complexity of the computations. This idea was stressed by R. Feynman (1982) who demonstrated that the problem of exponential complexity in terms of calculated probabilities can be reduced to a problem of polynomial complexity in terms of simulated probabilities. However, the main disadvantage of analog computers is a lack of universality. That is why the concept of a quantum computer became so attractive: its analog nature is based upon physical simulations of quantum probabilities, and, at the same time, it is universal (at least, for modeling physical world).

Although the development of the quantum-mechanical device is still in progress (Q.A. Turchette, 1995) a new quantum theory of computations has been founded (D. Deutsch, 1989: P. Shor, 1994). This theory suggests that there is a second fundamental advantage of the hypothetical quantum computer which is based upon the wave properties of quantum probabilities: a single quantum computer can follow many distinct computational paths all at the same time and produce a final output depending on the interference of all of them. This particular property opened up a new chain of algorithms which solve in polynomial time such hard problems as factorization and discrete log i.e., the problems which are believed to be intractable on any classical computer.

Thus, there are at least two areas where the quantum computer is expected to be superior over the classical one: quantum mechanics (due to simulation of quantum probabilities), and some specific combinatorial problems linked to operation research (due to interference of quantum probabilities).

However, besides of quantum mechanics, there are other computational "worlds" (biological, psychological, and social dynamics, informatics, artificial intelligence) where quantum mechanical micro structure has not been observed, Will the quantum computer be superior in these areas too? There are some doubts about that. Indeed, quantum mechanics is fully reversible, while the dynamics associated with biology, psychology and other intellectual activities is fundamentally irreversible: their time evolution describes transformations to higher levels of complexity. Consequently, quantum computer can provide only calculations rather than simulations in that areas.

Similar disadvantage may occur even in computations in classical physics if the models involve transport phenomena (heat transfer, diffusion, etc.) since then the problems also become irreversible. It should be noticed that despite the quantum-mechanical micro-structure of the transport phenomena, the transformation from the fully reversible Hamiltonian models to their microscopically irreversible versions is very sophisticated (in particular, it includes transition through chaotic instability, averaging, etc.). That is why direct simulations of classical physics by quantum computer is in question. In addition to that, computations in classical physics are expected to produce deterministic answers while quantum computer provides the answers in terms of probabilities. It should be noticed however that the last property of quantum computer can be useful for classical computing in case of Monte-Carlo approach if the quantum computer is exploited as a random number generator.

This paper has been motivated by an attempt to incorporate into classical computing the basic ideas of quantum computing: The simulations of

probabilities, and the interference between different branches of probabilistic scenarios.

It has to" be emphasized that dynamical simulations of classical probabilities conceptually is more difficult than those in quantum mechanics since classical dynamics is fully deterministic (if initial conditions are known exactly). The only window from classical dynamics to the probabilistic world is through chaos where the source of stochasticity is in finite precision of initial conditions. As shown in the second section of this paper, the combination of chaos and non-Lipschitz dynamical systems (M. Zak, 1994, 1996) can generate finite-state Markov chains. The corresponding stochastic process is described by two types of equations: the first one implements simulations in the form of random solutions, while the second (which is of the Fokker-Planck type) describes the evolution of the probabilities. Coupling between these two types of equations implements interference of probabilities similar (but not identical) to those in quantum mechanics.

The third, fourth and fifth sections describe a mathematical formalism behind the dynamical simulations of stochastic processes including probabilistic Turing machine, concept of stochastic attractors, as well as a new interpretation of simulated conditional probabilities.

The sixth section addresses applications to modeling of ntelligent systems.

In the seventh section, the effect of reducing computational complexity in combinatorial problems due to classical imitations of probability interference is

discussed. Based upon this effect, the concept of quantum intelligence is introduced.

The last section introduces a hypothetical deterministic dynamical system in a pseudo-euclidean space which, in principle, can simulate the Schrödinger equation.

2. Proof of Concept

Classical dynamics is fully deterministic if initial conditions are known exactly. Otherwise in some non-linear systems, small initial errors may grow exponentially so that the system behavior attains stochastic-like features, and such a behavior is called chaotic. The discovery of chaos contributed in better understanding of irreversibility in dynamics, of evolution in nature, and in interpretation and modeling of complex phenomena in physics and biology. However, there is a class of phenomena which cannot be represented by chaos directly. This class includes so called discrete events dynamics where randomness appears as point events, i.e., there is a sequence of random occurrences at fixed or random times, but there is no additional component of uncertainty between these times. The simplest example of such a phenomenon is a heartbeat dynamics which, in the first approximation, can be modeled by a sequence of pulses of equal heights and durations, but the durations of the pauses between these pulses are randomly distributed. Most processes of this type are associated with intellectual activities such as optimal behavior, decision making process, games, etc. In general, discrete events dynamics is characterized by a well-defined probabilistic structure of a piecewise-deterministic Markov chains, and it can be represented by probabilistic Turing machine. On the contrary, a probabilistic structure of chaos, and even the

appearance of chaos at all, cannot be predicted based only upon the underlying model without actual numerical runs. (The last statement can be linked to the Richardson's (1968) proof that the theory of elementary functions in classical analysis is undecidable). But is there a "missing link" between chaos and discrete events dynamics? And if it is, can this link be simulated based only upon physical laws without exploiting any man-made devices such as random number generators? A positive answer to this question would make a fundamental contribution to the reductionists view on intrinsic unity of science that all natural phenomena are reducible to physical laws. However, in addition to this philosophical aspect, there is a computational advantage in exploiting simulated probabilities instead of calculated ones in the probabilistic Turing machine: as shown by R. Feynman (1 982), the exponential complexity of algorithms in terms of calculated probabilities can be reduced to polynomial complexity in terms of simulated probabilities.

In this section we demonstrate that the missing link between chaos and a discrete event process can be represented by non-Lipschitz dynamics, (Zak, 1994, 1996.)

In order to illustrate the basic concepts of non-Lipschitz dynamics, consider a rectilinear motion of a particle of unit mass driven by a non-Lipschitz force:

$$\dot{v} = v v^{1/3} \sin \omega t, \quad v = const, \qquad [v] = \frac{m^{2/3}}{sec^{5/3}} \tag{1}$$

$$\dot{x} = v \tag{2}$$

where v and x are the particle velocity and position, respectively,

Subject to the zero initial condition

$$v = o \qquad \text{at} \qquad t=O \tag{3}$$

equation (1) has a singular solution

$$v = 0 \tag{4}$$

and a regular solution

$$v = \pm\left(\frac{4\nu}{3\omega}\sin^2\frac{\omega}{2}t\right)^{3/2} \tag{5}$$

These two solutions coexist at t = O, and this is possible because at this point the Lipschitz condition fails:

$$\left|\frac{\partial \dot{v}}{\partial v}\right|_{t\to 0} = \frac{1}{3}\nu v^{-2/3}\sin\omega t_{\,t\to 0} \to \infty \tag{6}$$

Since

$$\frac{\partial \dot{v}}{\partial v} \rangle\ 0 \qquad \text{at} \quad |v| \neq 0, \qquad t \rangle 0 \tag{7}$$

the singular solution (4) is unstable, and the particle departs from rest following the solution (5). This solution has two (positive and negative) branches [since the power in (5) includes the square root], and each branch can be chosen with the probability p and (1-p) respectively. It should be noticed that as a result of (5), the motion of the particle can be initiated by infinitesimal disturbances (such motion never can occur when the Lipschitz condition holds: an infinitesimal initial disturbance cannot become finite in finite time).

Strictly speaking, the solution (5) is valid only in the time interval

$$0 \le t \le \frac{2\pi}{\omega} \tag{8}$$

and at $t \le 2\pi / \omega$ it coincides with the singular solution (4)

For t) $2\pi / \omega$ equation (4) becomes unstable, and the motion repeats itself to the accuracy of the sign in equation (5).

Hence, the particle velocity υ performs oscillations with respect to its zero value in such a way that the positive and negative branches of the solution (5) alternate randomly after each period equal to $2\pi / \omega$.

Turning to equation (2), one obtains the distance between two adjacent equilibrium position of the particle:

$$x_i - x_{i-1} = \pm\int_0^{2\pi/\omega}\left(\frac{4v}{3\omega}\sin\frac{\omega}{2}t\right)^{3/2} dt = 64(3\omega)^{-5/2} v^{3/2} = \pm h \tag{9}$$

Thus, the equilibrium positions of the particle are

$$x_0 = 0, \qquad x, = \pm h, \qquad X_2 = \pm h \pm h... \tag{lo}$$

while the positive and negative signs randomly alternate with probabilities p and (1-p), respectively.

Obviously, the particle performs an unrestricted random walk: after each time period

$$\pi = \frac{2\pi}{\omega} \tag{11}$$

it changes its value on $\pm h$ [see equation (1 O)].

The probability density f(x,t) is governed by the following difference equation:

$$f(x,\ t\ +\ \tau) = pf(x\ -\ h,t) + (1-p)f(x\ +\ h,t) \tag{12}$$

which represents a discrete version of the Fokker-Planck equation, while

$$\int_{-\infty}^{\infty} f\,(x,t)\,dx\ =\ 1 \tag{13}$$

Several comments to the model (1) and its solution have to be made.

Firstly, the “viscous” force

$$F = -\nu v^{1/3} \tag{14}$$

includes static friction (see Eq. 6) which actually causes failure of the Lipschitz condition. These type of forces are well-known in theory of visco-plasticity (H. Ziegler, 1963). It should be noticed that the power 1/3 can be replaced by any power of the type:

$$k = \frac{2n-1}{2n+1},\ \ n = 1,2,\ldots \text{ etc} \tag{15}$$

with the same final result (12). In particular, by selecting large n, one can make k close to 1, so that the force (13) will be almost identical to its classical counterpart

$$F_c = -\nu v \tag{16}$$

everywhere excluding a small neighborhood of the equilibrium point V=o, while at this point

$$\frac{dF}{dv} \to \infty, \qquad \text{but} \left|\frac{\partial F_c}{\partial v}\right| \to 0 \qquad \text{at} \qquad v \to 0 \tag{17}$$

Secondly, without the failure of the Lipschitz condition (6), the solution to Eq. (1) could not approach its equilibrium v = O in finite time, and therefore, the paradigm leading to random walk (12) would not be possible.

Finally, we have to discuss the infinitesimal disturbances mentioned in connection with the instability of the solutions (5) at v = O. Actually the original equation should be written in the form:

$$\dot{v} = \nu v^{1/3} \sin\omega t + \varepsilon(t), \varepsilon \to O \tag{18}$$

where $\varepsilon(t)$ represents a time series sampled from an underlying stochastic process representing infinitesimal disturbances. It should be emphasized that this process is not driving the solution of Eq. (18): it only triggers the mechanism of instability which controls the energy supply via the harmonic oscillations $\sin\omega t$. As follows from Eq. (18), the function $\varepsilon(t)$ can be ignored

when $\dot{\upsilon}$ = O or when $\dot{\upsilon} \neq 0$, but the equation is stable, i.e. $\upsilon = \pi\omega, 2\pi\omega, \ldots$ etc. However, it becomes significant during the instants of instability when $\dot{\upsilon}$ = O at t = O, $\pi / 2\omega$ etc. Indeed, at these instants, the solution to Eq. (1) has a choice to be positive or negative if ε = O, (see Eq. (5)). However, with $\varepsilon \neq 0$,

$$sign\, x = sign\, \varepsilon \text{ at } (=0,\ \pi / 2\omega, \ldots \text{etc} \tag{19}$$

i.e., the sign of ε at the critical instances of time (19) uniquely defines the evolution of the dynamical system (18). Thus, the dynamical system (18) transforms a stochastic process (via its sample $\varepsilon(t)$) into a binary time series which, in turn, generates a random-walk-paradigm (18). Actually the solution to Eq.(18) represents a statistical signature of the stochastic process c .

Within the framework of dynamical formalism, the time series $\varepsilon(t)$ can be generated by a fully deterministic (but chaotic) dynamical system. The simplest of such system is the logistic map which plays a central role in population dynamics, chemical kinetics and many other fields. In its chaotic domain

$$y_{n+1} = 4y_n(1 - Y_n) > y_0 = 0.2 \tag{20}$$

the power spectrum for the solution is indistinguishable from a white noise. However, for the better match with Eq. (1 8), we will start with a continuous version of (20) represented by the following time-delay equation.

$$y(t + \tau) = 4y(t)[1 - y(t)],\ \tau = \frac{\pi}{2\omega} \tag{21}$$

$$y(t^*) = 0.2, \quad -\frac{\pi}{4\omega} (t^* < \frac{\pi}{4\omega} \tag{22}$$

The solution to Eq. (21) at t=O, $\pi / 2\omega$,... etc, coincides with the solution to Eq. (20), but due to the specially selected initial condition (22), the solution to Eq. (20) changes its values at $t = -\frac{\pi}{4\omega}, \frac{\pi}{4\omega}$..... etc, so that at the points t=o, $\pi / 2\omega$,... , the sign of this solution is well-defined,

Now assume that

$$\varepsilon(t) = \varepsilon_0 (y(t) - 0{,}51), \ \varepsilon_0 ((1. \tag{23}$$

The subtraction from y(t) its mean value provides the condition

$$p = 1 - p = \frac{1}{2} \tag{24}$$

Indeed, for the first hundred points in (23),

$$Sign\ \varepsilon = \begin{array}{l} --+-+++--+--+------+++--+ \\ --+-+-+-+--++----+-+---++-+ \\ +-+++-+++--++++++-++++++--+ \\ +----++---+-+---+--- \end{array} \tag{25}$$

has equal number of positive and negative values which are practically not correlated. Therefore, the statistical signature of the chaotic time series (23) is expressed by the solution to Eqs (12), (13) at $p = \frac{1}{2}$ with the initial conditions

$$f(0,0) = 1, \ f(x,0) = O \text{ if } x \neq O \tag{26}$$

which is a symmetric unrestricted random walk:

$$f(x,t) = C_n^m 2^{-n}; \quad m = \frac{1}{2}(n + x);\ n = \text{integer}\left(\frac{2\omega t}{\pi}\right) \tag{27}$$

Here the binomial coefficient should be interpreted asO wheneverm is not an integer in the interval $[0, n]$, and n is the total number of steps.

The connection between the solution (26) and the solutions to the system (18), (21), (2) should be understood as follows. Suppose we solve the system (1 8), (21), (2) subject to the initial condition (22) with $\upsilon =$ O and $x = 0$ *at* $t = 0$.

Since Eq. (21) is supersensitive to inevitable errors in (22), the solution will form an ensemble of chaotic time series, and for any fixed instant of time this ensemble will have the corresponding probability distribution which coincides with (26). In other words, the probabilities described by Eq. (12), are simulated by the dynamical system (18), (21) and (2) without an explicit source of stochasticity (while the "hidden" source of stochasticity is in finite precision of the initial condition (22)).

Combining several dynamical systems of the type (18), (21), (2) and applying an appropriate change of variables, one can simulate a probabilistic Turing machine which transfers one state to another with a prescribed transitional probabilities. Non-Markovian properties of such a machine can be incorporated by introducing time-delay terms in Eq. (2).

$$\dot{x} = \upsilon(t) + \alpha_1 \upsilon(t - \tau_0) + \alpha_2 \upsilon(t - 2\tau_0) + \ldots. \tag{28}$$

However, there is a more interesting way to enhance the dynamical complexity of the system (18), 21), (2). Indeed, let us turn to Eq. (23) and introduce a feedback from Eq. (2) to Eq. (18) as following:

$$\varepsilon = \varepsilon_0(y - x),\ \varepsilon_0 \ ((\ 1, \dot{y} = y - 0.\ 51 \tag{29}$$

Then the number of negative (positive) signs in the string (25) will prevail if x) O (x (o) since the effective zero-crossing line moves down (up) away from the middle. Thus, when (x = o) at t = O, the system starts with an unrestricted random walk as described above, and $|x|$ grows. However, this growth changes signs in Eq. (23) such that $\dot{x}$ (O if x) O, and $\dot{x}$) O if x (O. As a result of that

$$\mathrm{X}_{max} \leq \mathrm{y}_{max}, \quad \text{'rein} \geq \mathrm{y}_{min} \tag{30}$$

where y_{max} and y_{min} are the largest and the smallest values in the time series y (t), respectively. Hence, the dynamical system (18), (23), (2) simulates a restricted random walk with the boundaries (30) implemented by the dynamical feedback (29), while the probability

$$p(sign\varepsilon\rangle 0) = \begin{cases} 0 \text{ if } x \geq y_{max} \\ 1 \text{ if } x \leq y_{min} \end{cases} \tag{31}$$

For the sake of qualitative discussion, assume that p change linearly between x = y_{min} and x = y_{max}, i.e.,

$$P = \begin{cases} 0 & if\ x > y_{max} \\ \dfrac{y_{max} - x}{y_{max} - y_{min}} & if\ y_{min} \leq x \leq y_{max} \\ 1 & if\ x < y_{min} \end{cases} \tag{32}$$

(the actual function p(x) depends upon statistical properties of the underlying chaotic time series y(t). In particular, for the logistic map (20), small deviations from (32) take place only around the ends (i.e., when $x \cong y_{max}$ or $x \cong y_{min}$).

Then the simulated restricted random walk as a solution to Eqs. (12) and (32.)

Let us modify the feedback (29) as

$$\varepsilon = \varepsilon_0 [y - (x^2 - x)] \tag{33}$$

Now when x=O at t=O, the system is unstable since

$$sgn x = sgn \dot{x}, \quad -\infty < x < \frac{1}{2}, \tag{34}$$

and the process is divided into two branches. The negative branch (with the probability 1/2) represent an unrestricted random walk ($x \to 00$), while the positive branch (with the same probability ½) is eventually trapped within the basin of the attractor x = 1 since

$$sgn x = -sgn x, \quad \frac{1}{2} < x < \infty \tag{35}$$

simulating a restricted random walk as those described above with the only difference that its center is shifted from x = O to x = 1.

As a next step in complexity, introduce the information H associated with the random walk process described by Eqs. (12), (13):

$$H = -\int_{-\infty}^{\infty} f \log_2 f \, dx \tag{36}$$

and modify the feedback (29) as following:

$$\& = \varepsilon_0 \left[y - x(1 + H) \right] \tag{37}$$

Following the same line of argumentation as those performed for the feedback (29), one concludes that the feedback (38) becomes active only if the process is out of the domain of the maximum information, and therefore, it is always attracted to this domain.

Since Eq. (31) is still valid, we will apply the approximation similar to (32):

$$p = \begin{cases} 0 \; if \; x(1+H) \geq y_{max} \\ \dfrac{y_{max} - x(1+H)}{y_{max} - y_{min}} \\ 1 \; if \; x(1+H) \leq y_{min} \end{cases} \tag{38}$$

in order to continue our qualitative analysis. It should be noticed that now p depends not only on x, but also on f, and that makes Eq. (12) nonlinear. In addition to that, the system (18), (2) and (37), which is simulating probabilities, is coupled with the system (12), (13) and (38) describing the evolution of calculated probabilities. Actually due to this coupling, the entire dynamical

system attains such a self-organizing property as to maximize the information generated by the random walk,

The self-organizing properties of the system (18), (2), (37), (12), (13) and (38) mentioned above have a very interesting computational interpretation: they provide a mutual influence between different branches of probabilistic scenarios. Such an influence or interference, is exploited in hypothetical quantum computer (P. Shor, 1996) as a more powerful tool in a complexity theoretic sense, than classical probabilistic computations. However, in quantum computer, the interference is restricted to a linear unitary matrix transformation of probabilities (which is the only one allowed by quantum mechanics laws), while in the classical system (1 8), (2), (37) there is no such restriction; by choosing an appropriate probabilistic term in the feedback (37), we can provide an optimal interference. The price paid for such a property is the necessity to exploit the calculated probabilities (1 2), (13) and (38), and the consequences of that will be discussed in the last section.

3. Attractors, and Nonlinear Waves of Probability.

Let us return to the simplest case of a restricted random walk described by Eqs. (18), (29) and (2), and analyze the corresponding probability equation following from (12) and (32)

$$f(x,t+\tau)=\frac{y_{max}-x}{y_{max}-y_{min}}f(x-h,t)+\frac{x-y_{min}}{y_{max}-y_{min}}f(x+h,t) \qquad (39)$$

$$y_{min}\le x\le y_{max} \qquad (40)$$

For $x = y_{max}$

$$f(y_{max}, t+\tau) = f(y_{max}, +h, t) \tag{41}$$

Hence, if

$$f(y_{max} + h, t) = 0 \tag{42}$$

then

$$f(y_{max} t + \tau) = 0 \tag{43}$$

This means that with the initial conditions (26),

$$f(y_{max}, \tau) \equiv 0 \tag{44}$$

For the same reason:

$$f(y_{min}, \tau) \equiv 0 \tag{45}$$

Thus, if the solution starts within the interval (40), it is trapped there.

Let us assume now that instead of (26). the initial conditions are:

$$f(x_o, 0) = 1, \; f(x, 0) = 0 \; \text{if} \; x \neq 0, \; x_o > y_{max} \tag{46}$$

Then Eq (39) reduces to

$$f(x,t+\tau) = f(x+h,t),\ x \geq y_{max} \tag{47}$$

The solution to this equation can be written in the form of a traveling wave of the delta-function:

$$f = \delta(x - x_o + t),\quad x \geq y_{max} \tag{48}$$

moving toward the interval (40).

When x approaches y_{max}, Eq. (47) must be replaced by Eq. (39), and the solution is trapped again within the interval (40).

The same effect occurs if the solution starts with $x_o < y_{min}$.

Thus, the interval (40) represents a stochastic attractor (or a closed set of states) i.e. a stochastic process to which all the solutions of Eqs. (12), (32) converge irrespective of their initial conditions. Obviously that the intervals:

$$\infty > x \leq y_{max} \quad \text{and} \quad y_{min} > x > -\infty \tag{49}$$

represent the basin of this attractor.

The feedback (29) can be modified as:

$$\varepsilon = \varepsilon_o(\tilde{y} - ax), \qquad \alpha > 0 \tag{50}$$

without any qualitative changes in the results described above if the interval (40) is replaced by

$$\frac{y_{min}}{\alpha} \le x \le \frac{y_{max}}{\alpha} \tag{51}$$

However, there are significant qualitative changes if a < 0: the interval (51) becomes a stochastic repeller, i.e., any solution originated within this interval will eventually escape it.

In case of a nonlinear feedback (instead of (50)):

$$\varepsilon = \varepsilon_o \left[\tilde{y} - \varphi(x)\right] \tag{52}$$

the conditions

$$\frac{d\varphi}{dx} > 0 \quad or \quad \frac{d\varphi}{dx} < 0 \tag{53}$$

within the interval

$x' \le x \le x''$, where $\varphi(x') = y_{max}$, $\varphi(x'') = y_{min}$ (54)

lead to stochastic attractor, or repeller, respectively. Several stochastic attractors and repellers within the interval (54) are possible if the sign of the derivative $\partial\varphi / \partial x$ is changing.

Let us turn now to non-Markov stochastic processes when the probability of transition to another state depends not only upon the present, but also upon the past states. This properly can be easily simulated by modifying the feedback (52) as follows:

$$\varepsilon = \varepsilon_o \tilde{y} - \varphi[x(t), x(t-\tau), x(t-2\tau), \ldots\ldots etc] \quad (55)$$

In order to apply Eq. (12), the approximation (32) should be replaced by:

$$P = \begin{cases} 0 & if\ \varphi > y_{max} \\ \dfrac{y_{max} - \varphi}{y_{max} - y_{min}} & if\ y_{min} \le x \le y_{max} \\ 1 & if\ \varphi \le y_{min} \end{cases} \quad (56)$$

and substituted in (12)

Now Eq. (12) represents a more correlated (non-Markov) stochastic process, while by appropriate selection of the function φ in (55), a prescribed probability distributions as well as correlation's functions can be incorporated into the simulated stochastic process.

For illustration of dynamical simulation of a non-Markov stochastic process assume that

$$\tau = 1, \varphi = \bar{x}(t) + \bar{x}(t-1) - \frac{1}{2}, \quad \bar{x}(0) = \frac{1}{4}, \; y_{max} = |y_{min}|, \; \bar{x} = \frac{x}{|y_{max}|} \quad (57)$$

The last assumption is natural for the mean-zero chaotic process $\tilde{y}$ (see Eq. (29)).

Then

$$p=\begin{cases}0 \text{ if } \bar{x}(t)+\bar{x}(t-1)=\dfrac{1}{2}\\ 1 \text{ if } \bar{x}(t)+\bar{x}(t-1)=-\dfrac{1}{2}\end{cases} \qquad \textit{(58) \& (59)}$$

i.e.

$$p=\begin{cases}0 \quad \text{if } \bar{x}=\frac{1}{2}-\frac{1}{4}\cos\pi t\\ 1 \text{ if } \bar{x}=-\frac{1}{2}+\frac{3}{4}\cos\pi t\end{cases} \qquad \textit{(60) \& (61)}$$

'Thus, the non-Markov process characterized by the feedback (55), (57) has a stochastic attractor with a variable length:

$$0\le \ell=\tfrac{1}{2}-\tfrac{1}{4}\cos\pi t-\left(\tfrac{1}{2}+\tfrac{3}{4}\cos\pi t\right)=1-\cos\pi t\le 2, \qquad (62)$$

At the instants

$$t=0,2\pi,\ldots..etc \qquad (63)$$

the attractor disappears, and the process becomes fully deterministic.

One can verify that if φ in (57) is changed to the following:

$$\varphi=\bar{x}(t)+\bar{x}(t-1)+\tfrac{1}{2} \qquad (64)$$

then the domain (62) represents a stochastic repeller

An important step toward a higher complexity of simulated stochastic processes is dependence of the feedback (55) upon the probability functional (see, for instance, Eq. (37)). There are two fundamentally new effects here. Firstly, the equation (12) becomes nonlinear, and secondly, the dynamical simulations become explicitly dependent upon the probabilities.

In order to illustrate this, consider the following feedback instead of (50):

$$\varepsilon = \varepsilon_o \left[\tilde{y} - \frac{Sgn\, x}{\alpha\, y_{\max}} \int_{-\infty}^{\infty} x^2 f(x) dx \right], \qquad \alpha > 0 \tag{65}$$

where the variance

$$\sigma^2(t) = \int_{-\infty}^{\infty} x^2 f(x) dx \tag{66}$$

is a function of t and a functional of $f(x)$. Following the same line of argumentation as those applied for the derivation of Eqs. (40) and (51), one obtains that

$$\sigma^2 \le \alpha\, y_{\max}^2 \quad if \left| y_{\max} \right| = \left| y_{\min} \right|. \tag{67}$$

In other words, the feedback (65) provides such a stochastic attractor whose variance is limited by the condition (67). However, the lengths of this attractor can be found only from the solution to Eq. (12) with p expressed as:

$$p = \begin{cases} 0 \; if \; \sigma^2 \geq \alpha y_{max}^2, & x > 0 \\ 1 \; if \sigma^2 \geq \alpha y_{max}^2, & x < 0 \\ \frac{1}{2}\left(1 - \frac{Sgnx}{\alpha y_{max}^2}\sigma^2\right), & \sigma^2 \leq \alpha y_{max}^2 \end{cases} \qquad (68)$$

In the same way one can implement other limitations imposed upon probability densities of the simulated stochastic process.

In order to clarify the aspect of interference of probabilities provided by dependence of dynamical simulations upon probability densities, let us turn to Eqs. (2),(18) and (65). The solution to this system is random; however, each probabilistic scenario is controlled by the feedback (65) which includes the probability density, This probability density is governed by Eqs. (12) and (68) coupled with Eqs. (2), (18) and (65). Hence, the mechanism of interference of probabilities here is different from those in quantum mechanics: it is "artificially" organized via a special architecture of the dynamical system.

4. Simulation of Conditional Probabilities.

In the previous sections we dealt with only one state variable x (while v and y played the role of auxiliary variables). Now we will consider multi-dimensional system. Such systems could be simply obtained by starting with n identical systems of the type (18) and (2):

$$\dot{v}_i = v_i y_i^{\frac{1}{3}} \sin \omega t + \varepsilon_i(x_i), \dot{x}_i = v_i, i = 1,2, \ldots ..etc. \qquad (69)$$

and then changing variables

$$w_i = \varphi_i(v_i, \ldots v_n); \tag{70}$$

However, as will be shown below, there are such multi-dimensional systems which are more complex than those represented by Eq. (69),(70), and therefore, they should be discussed separately. In order to demonstrate this, consider the following two dimensional system:

$$\dot{v}_1 = \nu v_1^{\frac{1}{3}} \sin \omega t + \varepsilon_o \left(y - x_2 \right) \dot{x}_1 = v_1 \tag{71}$$

$$\dot{v}_2 = \nu v_2^{\frac{1}{3}} \sin \omega t + \varepsilon_o \left(\tilde{y} - x_1 \right), \quad \dot{x}_2 = v_2, \tag{72}$$

with respect to variables x_1 and x_2

Eqs. (71) and (72) are coupled via the feedbacks, but their associated probability equations are not coupled:

$$f_1(x_1, x_2, t+\tau) = p_1(x_2) f_1(x_1 - h, x_2, t) + \left[1 - p_1(x_2)\right] f_1(x_1 + h, x_2, t) \tag{73}$$

$$f_2(x_1, x_2, t+\tau) = p_2(x_1) f_2(x_1, x_2 - h, t) + \left[1 - p_2(x_1)\right] f_2(x_1, x_2 + h, t) \tag{74}$$

where

$$p_1 = \begin{cases} 0 \ if \ \bar{x}_2 > 1 \\ \frac{1}{2}\left(1 - \bar{x}_2\right) \ if \ \left|\bar{x}_2\right| \le 1, \\ 1 \ if \ \bar{x}_2 < 1 \end{cases} \qquad P2 \begin{cases} 0 \ if \ \bar{x}, \ge 1 \\ \frac{1}{2}\left(1 - \bar{x}_1\right) \ if \ \left|\bar{x}_1\right| \le 1 \\ 1 \ if \ \bar{x}_1 < 1 \end{cases} \tag{75}$$

$$|y_{max}|=|y_{min}|, \quad \bar{x}_1 = \frac{x_1}{|y_{max}|}, \quad X_{X2} = \frac{?}{|y_{max}} \tag{76}$$

It should be noted that x_2 and x. enter as parameters into Eqs. (73), and (74) respectively. That is why $f_1(x_1|x_2)$ *and* $f_2(x_2|x_1)$ represent conditional probability densities: f_1 describes the density of x_1 given x_2 and f_2 describes the density of X_2 given xl.

The solution to Eqs. (73) and (74) subject to the initial conditions (26), and the condition (13) are:

$$f_{1=}C_n^{m_1}\left(\frac{1-\bar{x}_2}{2}\right)^{m_1}\left(\frac{\bar{x}_2-1}{2}\right)^{n-m_1}, \quad m_1=\tfrac{1}{2}\left(n+\bar{x}_1\right), n=\text{integer}\left(\frac{2\omega t}{\pi}\right), \tag{77}$$

$$f_2 = C_n^{m_2}\left|\frac{1-\bar{x}_1}{2}\right)^{m_2}\left(\frac{\bar{x}_1-1}{2}\right)^{n-m_2}, \qquad m_2=\tfrac{1}{2}\left(n+\bar{x}_2\right) \tag{78}$$

Each of them represents a non-symmetric restricted random walk.

Now the following question can be asked: how to find an underlying joint probability density $\Phi(x_1x_2)$? It turns out that this is a hard question even from a conceptual viewpoint. Indeed, the relationships between $f_1(x_1|x_2), f_2(x_2|x_1)$ and $\Phi(x_1,x_2)$ are the following:

$$\Phi(x_1, x_2) \int_{-\infty}^{\infty} \Phi(z, x_2) dz = f_2(x_2|x_1) \int_{-\infty}^{\infty} \Phi(x_1, z) dz$$

whence

$$\frac{f_1(x_1|x_2)}{f_2(x_2|x_1)} = \frac{\int_{-\infty}^{\infty} \Phi(x_1, z) dz}{\int_{-\infty}^{\infty} \Phi(z, x_2) dz}$$

i.e.

$$\ell n \frac{f_1(x_1|x_2)}{f_2(x_2|x_1)} = \ell n \int_{-\infty}^{\infty} \Phi(x_1, z) dz - \ell n \int_{-\infty}^{\infty} \Phi(z, x_2) dz$$

and therefore

$$\frac{\partial^2}{\partial x, \partial x_2} \ell n \frac{f_1(x_1|x_2)}{f_2(x_2|x_1)} \equiv 0 \tag{79}$$

Thus the existence of the joint probability density $\Phi(x_1, x_2)$ requires that the conditional probability densities must satisfy the compatibility equation (79). But it is easily verifiable that the solutions (77) and (78) do not satisfy this equation, i.e., they are incompatible:

$$ink(\varphi_1, \varphi_2) = \frac{\partial^2}{\partial x, \partial x_2} \ell n \frac{f_1}{f_2} \not\equiv 0 \tag{80}$$

At the same time, there is nothing wrong with these solutions since they describe two stochastic processes which can be implemented by dynamical simulations. Hence, the only conclusion which can be made is that the joint probability in this particular case does not exist! But how "particular" is this case? Based upon the degree of arbitrariness to which the feedbacks in the system (71)(72) can be set up, it is obvious that the incompatibility of the conditional probabilities is a rule rather than an exception. In other words, there is a class of coupled stochastic processes for which joint probability does not exist, and therefore, they are inseparable, i.e., there is no such transformation of variables which would break them down into independent components. Similar conclusion (but in a different context) was made by G. Casello (1992).

Let us modify the feedbacks in Eqs. (71) and (72) as follows:

$$\&l= \left[\tilde{\varepsilon}\text{`y-}(\alpha_{11}x_1+\alpha_{12}x_2)\right],\ \varepsilon_1 = \varepsilon_0\left[\lambda_{\cdot Y-}(\alpha_{21}x_1+\alpha_{22}\grave{x}_2\right],$$

$$A = Const > O,\ \alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}\neq O \qquad (81)$$

Then the probability equations (73) and (74) are replaced by

$$f_1(x_1,x_2,t+\tau)+p_1(x_1,x_2)f_1(x_1-h,x_2,t)+[1-p(x_1,x_2)]f_1(x_1+h,x_2,t) \quad (82)$$

$$f_2(x_1,x_2,t+\tau)\cdot p_2(x_1,x_2)f_2(x_1,x_2-h,t)+[1-p(x_1,x_2)]f_2(x_1,x+h,t) \qquad (83)$$

where

$$p_1 = \begin{cases} 0 \text{ if } \alpha_{11}\bar{x}_1 + \alpha_{12}\bar{x}_2 > 1 \\ \frac{1}{2}\left[1 - (\alpha_{11}\bar{x}_1 + \alpha_{12}\bar{x}_2)\right] \text{ if } \left|\alpha_{11}\bar{x}_1 + \alpha_{12}\bar{x}_2\right| \le 1 \\ 1 \text{ if } \alpha_{11}x_1 + \alpha_{12}x_2 < 1 \end{cases} \quad (84)$$

$$p_2 = \begin{cases} 0 \text{ if } \alpha_{21}\bar{x}_1 + \alpha_{22}\bar{x}_2 > \lambda \\ \frac{1}{2}\left[\lambda - (\alpha_{21}x_1 + \alpha_{22}x_2)\right] \text{ if } \left|\alpha_{11}x_1 + \alpha_{12}x_2\right| \le \lambda \\ 1 \text{ if } \alpha_{21}x_1 + \alpha_{22}x_2 < \lambda \end{cases} \quad (85)$$

Following the same line of argumentation as those applied for one-dimensional case (see Eqs. (39)-(45)), let us find conditions for existence of coupled stochastic attractors.

For

$$\bar{x}_1 = \bar{x}_1^0, \bar{x}_2 = \bar{x}_2^0 \quad (86)$$

where $\bar{x}_1^0$ *and* $\bar{x}_2^0$ are solutions to the system:

$$\alpha_{11}\bar{x}_1^0 + \alpha_{12}\bar{x}_2^0 = 1, \qquad \alpha_{21}\bar{x}_1^0 + \alpha_{22}\bar{x}_2^0 = \lambda \quad (87)$$

the conditions similar to (41) will be satisfied:

$$f_1\left(\bar{x}_1^0, \bar{x}_2^0, t + \tau\right) = f_1\left(\bar{x}_1^0 + h, \bar{x}_2^0, t\right) \quad (88)$$

$$f_2\left(\bar{x}_1^0, \bar{x}_2^0, t + \tau\right) = f_2\left(\bar{x}_1^0, \bar{x}_2^0 + h, t\right) \quad (89)$$

and therefore, if

$$f_1\left(\bar{x}_1^0 + h, \bar{x}_2^0, t\right) = 0 \text{ and } f_2\left(\bar{x}_1^0, \bar{x}_2^0 + h, t\right) = 0 \tag{90}$$

then

$$f_1\left(\bar{x}_1^0, \bar{x}_2^0, t + \tau\right) \equiv 0, \quad f_2\left(x_1^0, x_2^0, t + \tau\right) \equiv 0 \tag{91}$$

"This means that with the initial conditions (26) (applied for both of Eqs. (82) and (83)),

$$f_1\left(\bar{x}_1^0, \bar{x}_2^0, t\right) \equiv 0, \quad f_2\left(\bar{x}_1^0, \bar{x}_2^0, t\right) \equiv 0 \tag{92}$$

For the same reason:

$$f_1\left(\bar{x}_1^0, \bar{x}_2^0, t\right) \equiv 0 \quad f_2\left(\bar{x}_1^0, \bar{x}_2^0, t\right) \equiv 0 \tag{93}$$

where $\bar{x}_1^{oo}$ *and* $\bar{x}_2^{oo}$ are solutions to the system:

$$\alpha_{11}\bar{x}_1^{oo} + \alpha_{12}\bar{x}_2^{oo} = -1, \; \alpha_{21}\bar{x}_1^{oo} + \alpha_{22}\bar{x}_2^{oo} = -\text{a} \tag{94}$$

Hence, as in the one-dimensional case (see Eqs. (49) and (45), the system may be trapped within the region:

$$\bar{x}_1^{oo} \le \bar{x}_1 \le \bar{x}_1^0, \quad \bar{x}_2^{oo} \le \bar{x}_2 \le \bar{x}_2^0 \tag{95}$$

if the inequalities following from (95)

$$\bar{x}_1^{\infty} \le \bar{x}_1^0, \bar{x}_2^{\infty} \le \bar{x}_2^0 \tag{96}$$

also follow from the solutions to the systems (87) and (94), and that imposes certain constraints upon the coefficients $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}$ and λ.

Indeed, these solutions are:

$$x_1^0 = \frac{\alpha_{22} - \lambda\alpha_{12}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}} \qquad x_2^0 = \frac{\alpha_{11}\lambda - \alpha_{21}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}} \tag{97}$$

$$\bar{x}_1^{\infty} = -\frac{\alpha_{22} - \lambda\alpha_{12}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}} = -\bar{x}_1^0, \quad \bar{x}_2^0 = -\frac{\alpha_{11}\lambda - \alpha_{21}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}} = -\bar{x}_2^{\infty}$$

and therefore, the inequality (96) is satisfied if

$$\frac{\alpha_{22} - \lambda\alpha_{12}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}} > 0, \qquad \frac{\alpha_{11}\lambda - \alpha_{21}}{\alpha_{11}a_{22} - \alpha_{12}\alpha_{21}} > 0 \tag{98}$$

Thus, the inequalities (98) guarantee that if the initial conditions for the system (82), (83) are within the region (95), then the solutions will be trapped there. But if the initial conditions are outside of this region, then following the same line of argumentation as those for one-dimensional case, (see Eqs. (46)-(48)), one concludes that eventually the solutions will approach the region (95) and will remain there.

Hence, the inequalities (98) represent the necessary and sufficient conditions that the region (95) is a two-dimensional stochastic attractor.

It should be noticed that in contradistinction to the stochastic attractors introduced by Zak, (1996), this one is inseparable, i.e., it cannot be broken down into two independent one-dimentional stochastic attractors by any change of variables.

It can easily be verified that opposite signs in (98) convert the region (95) into stochastic repeller, while different signs

$$\frac{\alpha_{22} - \lambda\alpha_{12}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}} > 0, \qquad \frac{\alpha_{11}\lambda - \alpha_{21}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}} < 0 \tag{99}$$

lead to attraction in x_1 direction, and repulsion in x_2 direction, i.e., to a mixed type of a limit set.

If

$$\alpha_{22} = \lambda\alpha_{12}, \quad \alpha_{21} = \lambda\alpha_{11} \tag{loo}$$

then the region (95) degenerates into a point, while if the only one equality in (100) holds, then this region degenerates into a line $x_2 = \text{Const}$, or $x_1 = \text{Const}$, respectively.

A general form of multi-dimensional simulated conditional probabilities can be derived from Eqs. (71)-(75):

$$\dot{v}_i = \nu v_i^{\frac{1}{3}} Sin\omega t + \varepsilon_0\{\tilde{y} - \varphi_i[x_1,\ldots x_n, \Phi_i(f_1,\ldots f_n)], \quad \dot{x}_i = v_i \tag{101}$$

$$f_1(x_1,\ldots x_n, t+\tau) = p_i f_i(x_1,..,x_i - h,\ldots x_n, t) + (1-p_i) f_i(x_1,..,x_i + h,\ldots x_n t) \tag{102}$$

where

$$p_i = \begin{cases} 0 \text{ if } \varphi_i \geq y_{max} \\ \dfrac{y_{max} - \varphi_i}{Y_{max} - y_{min}} \text{ if } y_{min} \leq \varphi_i \leq y_{max} \\ 1 \text{ if } \varphi_i \leq y_{min} \end{cases} \tag{103}$$

and $\varphi_i(f_1,\ldots f_n)$ are functional of the probabilities $f_1,\ldots\ f_n$ Non-Markovian effects can be incorporated in this system by including time-delays variables $x_i(t-\tau)$ as additional arguments for the functions φ_i.

It should be emphasized that as in the two-dimensional case, all the stochastic processes here, in general, are inseparable,

5. Simulations of Probabilistic Turing Machine

All the simulated stochastic processes discussed above had the following limitation: they could perform a random jump only to the adjacent neighboring states, thereby representing a simple random walk. However, for the purpose of universal computations, such a performance is not sufficient: random jumps from an arbitrary state to any other state with prescribed probability is required. In this section we will demonstrate how to modify the models introduced above to attain such a property.

Let us turn to Eqs. (21) and (22) and assume that

$$\tau = \frac{\pi}{4\omega} n_1,\ \frac{\pi}{4\omega} n_2,\ldots etc.,\quad -\frac{\pi}{4\omega} n < t^* < \frac{\pi}{4\omega} \tag{104}$$

Now the solution to Eq. (20) changes its values at

$$t = \frac{\pi}{4\omega} n_1, \frac{\pi}{4\omega} n_2, \cdots etc \tag{105}$$

This means that the solution to Eqs. (18), and (2) cannot have random jumps between the intervals (105), i.e., the length of a step in the random walk will be

$$h_n = nh \tag{106}$$

where h is expressed by Eq. (9).

Suppose that n is a random variable which is simulated by the dynamics:

$$\dot{\upsilon}_n = \nu_n \upsilon_n^{\frac{1}{3}} Sin\omega t + \varepsilon_0[\tilde{y} - \varphi(n,x)], \dot{n} = \upsilon_n \tag{107}$$

where φ is an arbitrary function of n and x.

Then, analogously to Eq. (9):

$$n_i^{-} n_i = 64(30.))^{-5/2} \nu_n^{3/2} = \ell \tag{108}$$

where ℓ is the unit step of change of the variable n,

This step can be set

$$\ell = 1 \tag{109}$$

by an appropriate choice of v_n

$$v_n = \frac{(3\omega)^{5/3}}{16} \tag{110}$$

Hence,

$$n = \pm 1 \pm 1 \pm 1 \pm etc. \tag{111}$$

while the positive and negative signs in Eq. (1 11) alternate randomly with the probabilities p_n and $(1 - p_n)$, respectively, where

$$_n = \begin{cases} 0 \ \ if \ \ \varphi(n,x) > y_{\max} \\ \dfrac{y_{\max} - \varphi(n,x)}{y_{\max} - y_{\min}} \qquad if \ \ y_{\min} \le \varphi(n,x) \le y_{\max} \\ 1 \qquad\qquad\qquad if \ \ \varphi(n,x) < y_{\min} \end{cases} \tag{112}$$

The probability density $f_n(n,x,t)$ is governed by the following difference equation:

$$f_n(n,x,t+\tau) = p_n f_n(n-1,x,t) + (1-p_n) f_n(n+1,x,t) \tag{113}$$

$$\sum_{n=1}^{N} f_n(n,x,t) = 1 \tag{114}$$

where p_n is expressed by Eq. (1 12).

Thus, now in all of the equations considered above (see Eqs. (12), (39), (73), (74),(82),(83), the unit step of change in x, or xi is expressed by eq. (106) where n is a random variable with the probability distribution $f_n(n,x,t)$ governed by Eqs. (1 *13)* and (1 14). Therefore, all the equations listed above should be modified, and we will illustrate this modification based upon Eq. (39) which now reads:

$$f(x,t+\tau) = \sum_{n=1}^{N} f_n\left[pf(x - nh,t) + (1 - p)f(x + nh,t)\right] \tag{115}$$

$$\int_{-\infty}^{\infty} f(x,t)dx = 0 \tag{116}$$

where p and f_n are expressed by Eqs. (32) and (1 13),(1 14), respectively.

The corresponding dynamical system which simulates the probability equations (1 15) and (1 16) is:

$$\dot{\upsilon} = \nu\upsilon^{1/3}Sin\omega t + \varepsilon_0(\tilde{y} - x),\ x = \upsilon \tag{117}$$

$$\dot{\upsilon}_n = \nu_n\upsilon_n^{1/3}Sin\omega t + \varepsilon_0\left[\tilde{y} - \varphi\left(n_*, x\right)\right],\quad \dot{n}_* = \upsilon_{n_*},\ n = N - n_* > 0 \tag{118}$$

$$\tilde{y} = y(t) - 0.51, \tag{119}$$

$$y(t+\tau) = 4y(t)[1 - y(t)],\ \tau = \frac{\pi}{2\omega}n \tag{120}$$

$$y(t^*) = 0.2,\ \frac{\pi}{4\omega} < t^* < \frac{\pi}{4\omega} \tag{121}$$

Thus, the variable x simulated by Eqs. (1 17) performs a restricted non-symmetric, generalized random walk: after each time period $\tau = 2\pi / \omega$ it changes its value on $h_n = \pm nh$. But n, in turn, is also a random variable simulated by Eqs. (1 18), and its probability density follows from Eqs. (1 15) and (1 16). Hence, at each step, the variable x has a probability p or (l-p) to move right or left, respectively, and in a selected direction it has a probability p_n to move from any fixed point x =X..

Indeed, let us select a point x = x_o. This will uniquely define the probability $p(x_o)$ *(see* Eq. (32)) that the next step will be directed to the right. Utilizing Eqs. (1 13) and (1 14), one can find the probability $f_n(x_o, n_o)$ that the length of the jump will be

$$h_n = n_0 h \qquad (122)$$

Hence, the following transition probability matrix can be introduced:

$$P = \begin{vmatrix} p_{11} p_{12} \ldots p_{1N} \\ p_{21} p_{22} \ldots p_{2N} \\ \cdots\cdots\cdots \\ p_{N1} p_{N2} \ldots p_{NN} \end{vmatrix} \qquad (123)$$

where

$$p_{ij} = p f_n(x_i, n_j) \qquad (124)$$

is the probability that the system being in the state x_i would move to the right making the step of the length

$$h_n = n_j h \qquad (125)$$

Conversely,

$$p_T^* = \begin{pmatrix} p_{11}^* \cdots p_{1N}^* \\ \cdots\cdots \\ \cdots\cdots \\ p_{N1}^* \cdots p_{NN}^* \end{pmatrix} \qquad (126)$$

where

$$p_{ij}^* = (1-p) f_n(x_i, n_j) \qquad (127)$$

is the probability that the system being in the state( ) xi would move to the left making the step of the length (125).

Obviously, if the initial probability distribution over the states $x_1, x_2, \ldots x_N$ is $\alpha_k^{(1)}$:

$$\alpha_k^{(2)} = \alpha_j^{(1)} p_{jk}, \qquad \overset{*}{\alpha}_k^{(2)} = \alpha_j^{(1)} \overset{*}{p}_{jk} \qquad (128)$$

where $a_k^{(2)}$ and $\overset{*}{\alpha}_k^{(2)}$ are the probability distribution over the states to the right and to the left of the original states respectively.

The basic property of the relationships (128) is that the probability α_k assigned to the state x_k depends only upon transition probabilities p_{jk} to the same state from all other states, and it does not depend upon the transition probabilities $p_{j\ell}(\ell \neq k)$ to different states. Physically it means that if a certain branch of the probabilistic scenario is already chosen in the course of dynamical simulations, then all other branches become irrelevant: they will never effect the evolution of the dynamical system, In other words, there is no interference between classical probabilities, and therefore, one dynamical "device" can process only one Probabilistic branch.

simulations of classical and quantum probabilities; in particular, there is no way in which quantum probabilities can be dynamically imitated by a classical device.

Thirdly, in classical case the probability that a particular configuration is reached at a certain step k in the computation is the sum of the probabilities of all the nodes corresponding to that configuration at the level k in the computational tree.

In contradistinction to that, in QTM, the probability of the same configuration is the square of the sum of the amplitudes of all leaf nodes corresponding to that configuration. As a result of that, the probabilities interfere in such a way that two different probabilistic branches can amplify or cancel each other (the last case occurs if some of the probability amplitudes are negative). This means that in quantum computations one cannot follow a selected probabilistic branch as in the classical case: a QTM processes simultaneously all the probabilistic scenarios in the form of a special type of their superposition, and that particular property had been proven to be the most important in reducing the exponential complexity of computations to the polynomial one for some hard problems like factorization of large numbers, (P. Shot, 1996).

However, there is another effect of probability interference which can be associated with so called emergent computations, (S. Forrest, 1990). Emergent computations is an alternative to parallel computing which exploits the interactions among simultaneous computations to improve efficiency, increase flexibility, or provide a more natural representation. The basic idea behind the

emerging computations is that if a physical phenomenon is described in terms of its information processing properties, then the information which is absent at lower levels can exist at the level of collective activities. That is why the emerging computations can lead to effects of self -organizaton, cooperation between primitive components without global control.

It is interesting to notice that the property to perform emerging computations has not been well pronounced in theory of QTM, probably because of limitations imposed upon interactions between different branches of computations by unitary transformations required by quantum mechanics. As will be shown below, the emerging computations can be simulated within the framework of the Turing machine introduced above (see Eqs. (1 17)-(121), in a more flexible way; the only price paid for that will be the necessity to exploit a 'calculated" probilities. In order to demonstrate that, let us turn to Eq. (1 17) and modify it as follows:

$$\mathrm{v} = v\upsilon^{1/3} Sin\omega t + \varepsilon_o[y - \varphi(x, H, \sigma, \ldots)], x = \upsilon \tag{129}$$

where H and σ are given by expressions (36), and (66), respectively.

Then, Eq. (32) should be replaced by

$$p = \begin{cases} 0 \text{ if } \varphi(x, H, \sigma, \ldots) & \geq y_{max} \\ 1 \text{ if } \varphi(x, H, \sigma, \ldots) & \leq y_{min} \\ \dfrac{y_{max} - \varphi(x, H, \sigma, \ldots)}{y_{max} - y_{min}} \text{ if } y_{min} \leq \varphi(x, H, \sigma, \ldots) \leq y_{max} \end{cases} \tag{130}$$

and therefore, the components of the transition probability matrix (124) will depend upon H, σ, and other functional of the probability distribution $f(x,t)$. This means that now the evolution of a certain probabilistic branch simulated by Eqs. (1 17) - 121) will depend upon evolution of other branches via the functional H, σ,...etc., and therefore, it will be coupled with the equations (11 3)-(116) which govern the probabilities $f_n(n,x,t)$ and $f(x,t)$.

Thus, the system (1 13)-(1 16), (1 18)-(121) and (129) represents a classical Turing machine performing emerging computations, while the way in which different branches of computations interfere can be set up by an appropriate choice of functional in Eq. (29). However, it should be recognized that there is a difference between the role of Eqs. (1 13) - (1 16) in the case of Eq. (1 17), i.e., without interference, and in the case of Eq. (129), i.e., with the interference of probabilities. In the first case, the dynamical equations (1 17) - (121) simulate the probabilities f and f_n and therefore they simulate Eqs. (1 13) - (1 16). In the last case, Eqs. (1 18)-(129), (129) and Eqs. (1 13) - (1 16) are coupled, i.e., the probabilities cannot be simulated without equations describing "calculated" probabilities.

6. Simulation of Intelligent Systems

When dynamical systems describe natural phenomena, their basic properties are defined by energy and its time evolution. However, for those dynamical systems which model intellectual activities such as optimal behavior, decision making process, games, the more useful characteristic of the performance is information rather than energy.

Let us consider, for instance, a dynamical system (1 17) - (121). The evolution of the information stored in it is defined as:

$$H(t) = -\int_{-\infty}^{\infty} f(x,t)\log_2 f(x,t)dx \qquad (131)$$

while $f(x,t)$ is governed by Eqs. (1 15),(1 16), which are uniquely determined by the parameters of the system (117) - (121).

However, the fundamental difference between the systems modelling natural phenomena, i.e., physical systems, and man-made, or intelligent systems is not only in the way in which they are described, (as a matter of fact, physical systems can also be described in terms of information, H. Haken, 1988), the basic property of intelligent systems is that their structure, and therefore, the structure of the information (131), is not given in advance: it is supposed to be created based upon the purpose to which the system serves.

For illustration, consider the system (101) and present the functions φ_i in a parametrized form adopting a neural net formalism, (M. Zak, 1994):

$$\dot{v}_i = \nu v_i^{1/3} Sin\omega t + \varepsilon_o(\tilde{y} - \varphi_i),\ \dot{x}_i v_1,\ i = 1, 2\ldots n \qquad (132)$$

$$\varphi_i = \tanh\sum_{j=1}^{n} T_{ij}x_j \qquad (133)$$

where T_{ij} do not depend upon x_i (but they may depend upon the functional of

$f(x)$; the hyperbolic tangent tanh represents nonlinearities in $\varphi_i(x_{i,\ldots}x_n)$. Thus, the dynamical system (1 12) in terms of its probability evolution (102) is uniquely defined by the choice of the constants T_{ij}, and therefore, the problem of the synthesis of an intelligent system can be reduced to finding these constants based upon the objective of the system performance.

Let us assume that the purpose of the system performance is to minimize some functional of the probability distributions $f_i,\ldots\ f_n$:

$$\Phi(f_1,\ldots f_n)\rightarrow \min \tag{134}$$

where $f,$ depends upon T_{ij} via Eqs. (102),(103) and (133)

Then T_{ij} are found from the system:

$$\frac{\partial\Phi}{\partial T_{ij}}=0;\ \ i,j=1,2,\ldots\ \mathrm{n} \tag{135}$$

However, despite the conceptual simplicity of this approach, its practical significance is limited since in real life situations the global objective in the form (134) is not available. Instead, each dynamical unit, i.e., each variable xi , has its own local objective which can be partly compatible, and partly contradictory to local objectives of other units. Surprisingly, very often such systems exhibit very interesting properties associated with the concept of collective brain.

The concept of the collective brain has appeared recently as a subject of intensive scientific discussions from theological, biological, ecological, social,

and mathematical viewpoint. It can be introduced as a set of simple units of intelligence which can communicate by exchange of information without explicit global control. The objective of each unit may be partly compatible and partly contradictory, i.e., the units can cooperate or compete. The exchanging information may be at times inconsistent, often imperfect, non-deterministic, and delayed. Nevertheless, observations of working insect colonies, social systems, and scientific communities suggest that such collectives of single units appear to be very successful in achieving global objectives, as well as in learning, memorizing, generalizing and predicting, due to their flexibility, adaptability to environmental changes, and creativity.

Let us assume that each member (or a dynamical unit) characterized by the variable x_i, has its own version of the global objective of the whole dynamical system which can be expressed in the form similar to (134):

$$\Phi_i(f_{i,}\dots f_n) \rightarrow \min \tag{136}$$

while, in general,

$$\Phi_i \neq \Phi_j \; if \; i \neq j$$

Each unit can learn (in its own way) the global objective of the system during previous collective tasks. Based upon that, it may "derive" its own version of the system (135):

$$\frac{\partial \Phi_k}{\partial T_{ij}^{(k)}} \text{“ } 0, \qquad k = 1,2,\dots n \tag{137}$$

and therefore, its own version of the whole dynamical system in the form smilar to Eqs. (1 32):

$$\dot{\upsilon}_i^{(k)} = \nu\left(\upsilon_i^{(k)}\right)^{1/3} Sin\omega t + \varepsilon_o\left(\tilde{y} - \varphi_i^{(k)}\right), \dot{x}_i^{(k)} = \upsilon_i^{(k)}, \; i = 1,2,\ldots \text{n} \qquad (138)$$

$$k = 1,2,\ldots \text{n}$$

Here $\upsilon_i^{(i)}, x_i^{(i)}$ are the actual values of the variables, while $\upsilon_i^{(k)}, x_i^{(k)} (k \neq i)$ are the values of the same variables predicted by the K^{th} dynamical unit,

Hence, as a result of the collective brain paradigm, the oiriginal dynamical system (132) of 2n equations with respect to 2n variables Xi and υ_i is replaced by the system (138) of $2n^2$ equations with respect to $2n^2$ variables $x_i^{(k)}$ and $\upsilon_i^{(k)}$, $i,k = 1,2\ldots n$.

Since the last system has the same dynamical structure as the original dynamical system (132), its solution can be described by n^2 equations similar to (1 2 7) :

$$f_i^{(k)}\left(x_1^{(k)} \ldots x_n^{(k)}, \; t + \tau\right) = p_i^{(k)} f_i^{(k)}\left(x_1^{(k)} \ldots X_{i,(k)} - h, \ldots x_n^{(k)}\right) + \left(1 - p_i^{(k)}\right) f_1^{(k)}\left(x_1^{(k)}, \ldots x_i^{(k)} + h, \ldots x_n^{(k)} + h, \ldots x_n^{(k)}, t\right) \qquad (139)$$

where

$$p_i^{(k)} = \begin{cases} 0 \ if \ \varphi_i^{(k)} \geq y_{max} \\ \dfrac{y_{max} - \varphi_i^{(k)}}{y_{max} - y_{min}} \qquad if \ y_{min} \leq \varphi_i^{(k)} \leq y_{max} \\ 1 \qquad\qquad\qquad if \ \varphi_i^{(k)} \leq y_{min} \end{cases} \tag{140}$$

and

$$\varphi_i^{(k)} = \tanh \sum_{j=1}^{n} T_{ij}^{(k)} x_j^{(k)} \tag{141}$$

As follows from Eqs. (139)-(141), the dynamics with the collective brain is less predictable than the original dynamics. However, in contradistinction to the original dynamics which requires a global control for its performance, the last version of dynamics is more flexible: it can perform relatively well based upon the autonomy of the dyamical units which can predict the events if the actual values of the variables are not available.

The autonomy of the dynamics with collective brain can be increased if each unit can have not only its own version of the global objective of the system, but also its versions of the global objectives of others dynamical units. Clearly such an ability will require deeper correlations between the dynamical units which can be achieved by more intensive learning during the previous collective tasks. From the analytical viewpoint, the complexity of this dynamic system will be significantly higher: the system having the same stucture as Eqs. (138) and (139), will contain $2n^3$ equations with respect to $2n^3$ variables. In the same way one can introduce more autonomous (but more complex) dynamical systems with collective brain of higher dimensionalities.

The intelligent systems of the type (138), (139) or of its more complex versions discussed above, can be linked to game theory. Indeed, here each i^{th} player (represented by the corresponding variable xi), tries to achieve its local objective by taking into account the knowledge about possible local objectives of other players. However, in contradistinction to the classical game theory which is based upon 'calculated" probabilities, the intelligent dynamical systems discussed above are based upon simulated probabilities: they are capable of learning probabilistic strategies, and the knowledge is acquired and stored in the deterministic coefficients Tij.

7. Quantum Intelligence

In the previous sections we have analyzed an impact of ideas introduced in connection with a hypothetical quantum computer upon concepts of classical computing. Our attention was focused on simulation of classical probabilities and imitation of a quantum effect of probability interferences. Actually, the last effect has ignited the scientific community by its potential for powerful non-quantum applications to computational problems of exponential complexity. This kind of problem became an obstacle to progress in many classical areas such as operation research, artificial intelligence, combinatorial optimization, etc. Because of non-quantum nature of these applications, classical simulations which could imitate the probability interference can be considered as an attractive alternative to a hypothetical quantum device. Such simulations were introduced in the form of the dynamical system (1 12) - (121), and they can be associated with quantum intelligence.

Let us take a close look at this system. Actually it simulates the evolution of a classical probabilistic Turing machine. If this machine has N possible states, and it starts from an initial roof of the N^{th} levelled "decision" tree, there are 2^N possible probabilistic scenarios to approach the N^{th} level. By specifying a set of scenarios consequent to each possible "action," a 'decision" free facilitates evaluation of this action. Therefore, in order to evaluate all actions and select the best (based upon a prescribed objective), one has to perform an exhaustive search and run 2^N scenarios, while this number grows exponentially with the linear growth of the dimensionality N.

Let us assume now that there is an interference of these probabilistic scenarios. In our case such an interference is achieved by incorporating probability functional (such as $E(f), \sigma(f), H(f)$ etc) in to the dynamical equation (1 18). Then only a simple run of Eqs. (1 17)-(121) will include simultaneously all the contributions from other probabilistic branches via the probability functional, and due to these contributions the actual trajectory may jump from one branch to another. By selecting an appropriate structure of the feedback in Eq. (1 18), this trajectory can be optimized subject to a prescribed objective of the system performance.

The strategy for the trajectory optimization can be drawn out of the methodology proposed in the previous section. Indeed, consider the system (1 12)-(121) and Eq. (32), and assume that the objective function is expressed in the form of the maximum of the information H:

$$H \to \max, H = -\sum_{i=1}^{N} f(x_i) \log_2 f(x_i) \tag{142}$$

subject to the constraints:

$$\sum_{i=1}^{N} f(x_i) = {}_1 \sum_{i=1}^{N} x_i f(x_i) = E(t) \tag{143}$$

where the expectation E is a prescribed function of time.

Then the optimal probability distribution $f_0(x_i t)$ is found from the condition:

$$\frac{\partial}{\partial f}\left[H + \lambda_1 \sum_{i=1}^{N} f(x_i) + \lambda_2 \sum_{i=1}^{N} x_i f(x_i)\right] = 0 \tag{1 44}$$

whence

$$f_0(x_i t) = e^{\lambda_1 + \lambda_2 x_i - 1}, \quad \lambda_1 = \lambda_1(t), \lambda_2 = \lambda_2(t) \tag{145}$$

Here λ_1 and λ_2 are the Lagrange multipliers found from teh constraints (143). Now one has to find such a solution to Eqs. (1 15)-(1 16) which would have a minimal deviation (in the least square sense) from Eq. (145).

This means that for each time step τ:

$$\sum_{i=1}^{N} \left[\tilde{f}(x_i t + \tau) - f_0(x_i t + \tau)\right]^2 \to \min \tag{146}$$

where

$$\tilde{f}(x_i, t + \tau) = \sum_{n=1}^{N} f_n^0 \left[p f_0(x_i - nh, t) + (1 - p) f_0(x_i + nh, t)\right] \tag{147}$$

The minimum in (146) should be sought with respect to f_n^0 subject to the constraint (1 16). This leads to minimization of the following function:

$$\theta_f = \sum_{i=1}^{N} \left[\sum_{n=1}^{N} f_n^0 \Phi(x_i, n, t) - f, \right] + \lambda \sum_{n=1}^{N} f_n^o \to \min \qquad (148)$$

where λ is a Lagrange multiplier, and

$$\Phi(x_i, n) = p f_0(x_i - nh, t) + (1-p) f_0(x_i + nh, t) \qquad (149)$$

Parametrizing φ_n^o as functions of x:

$$f_n^0 = \sum_{k=1}^{m} T_{kn}^f x^m \qquad (150)$$

one reduces the problem to finding the constraints T_{kn}^f from the system of linear equations:

$$\frac{\partial \theta_f}{\partial T_{kn}^f} = 0 , \quad k = 1,2 \ldots m; \; n = 1,2 \ldots N \qquad (151)$$

which, together with Eqs. (1 50) define the optimal (subject to the objectives (142),(1 43) probability distributions, and therefore, the optimal transitional probabilities (124), for each time step τ.

The Lagrange multiplier λ is found from the constrain (1 16).

The next step is to find the appropriate coefficients p_n^o in Eq. (1 13) which provide the optimal distributions $f_n^o(x)$ determined above. Applying the same strategy as those for Eq. (1 15), one obtains:

$$\theta_p = \sum_{i=1}^{N} \left[\tilde{f}_n(n,x,t+\tau) - f_n^0(n,x,t+\tau) \right]^2 \rightarrow \min \tag{152}$$

where

$$\tilde{f}_n(n,x,t+\tau) = p_n^0 f_n^o(n-1,x,t) + \left(1-p_n^0\right) f_n^0(n+1,x,t) \tag{153}$$

The minimum in (152) is sought with respect to p_n^0.

Parametrizing p_n^0 as functions of x:

$$p_n^o = \sum_{k=1}^{m} \mathrm{T}_{kn}^p x^m \tag{154}$$

one again reduces the problem to finding the constraints T_{kn}^p from the system of linear equations:

$$\frac{\partial \theta_p}{\partial \mathrm{T}_{kn}^f} = 0 \tag{155}$$

which, together with Eq. (154), determine the optimal functions p_n^0 in Eqs. (1 12) and (1 13), and therefore, the optimal structure of the feedback in Eq. (1 18), for each time step τ :

$$(\mathrm{p}(\mathrm{n},\mathrm{x}) = y_{max} - p_n^o\left(y_{max} - y_{min}\right) \tag{156}$$

Thus, starting with the objective (142) subject to the constraints (143), for each time step τ one can determine the optimal feedback (1 55) for Eq. (1 18).

Now a single run of Eqs. (1 17)-(121) at each time step, will include "thinking," i.e., determining the structure of the feedback (156) for the reset time step τ via solving Eqs. (151) and (155).

It should be understood that the optimal trajectory discussed above has a certain meaning only in the probabilistic sense. Indeed, each run of Eqs. (1 17)-(121) will result in different optimal trajectory, but all of such trajectories will form an ensemble whose probabilistic properties are optimal subject to the objective (142), (143).

In order to evaluate the computational complexity of the optimization performed by quantum intelligence, we recall that solutions to Eqs. (151) and (1 55) (which implement 'thinking" accompanying the single run of eqs. (1 17) - (121)) have polynomial complexity, and therefore, the quantum intelligence paradigm eliminated combinatorial explosion.

Obviously the same approach can be applied to more complex intelligent systems where a global objective is replaced by a set of competing local objectives (see Eqs. (138)-(141). In these systems, quantum intelligence is implemented in the form of dependence of the feedbacks in Eqs. (138) on the probability functional via the coefficients $T_{ij}^{(k)}$ (see Eqs. (133) and Eqs. (137)). As a result of that, the system attains some new self-organizing properties which have not been prescribed in advance. The mechanism of such an emerging

phenomenon exploits contribution of the paradign of collective brain when each dynamical unit not only has its own local objective, but also predicts local objectives of other units.

So far, the interference of probabilities was implemented via the dependence of the feedback in the dynamical equations upon the probability functional $E(f), \sigma(f), H(f)$, etc., i.e. via the global influence between different probabilistic branches.

However, the interference of probabilities can be local if the feedback depends upon the probability f and its derivatives, but not upon their functional. Turning, for instance, to Eqs. (101), one can modify the feedback φ_i in the following way:

$$\varphi_i = \varphi_i\left(x_i, \ldots, f_n, f_n, \frac{\partial f_1}{\partial x}, \ldots \frac{\partial f_n}{\partial x}, \ldots etc \right) \tag{157}$$

For a qualitative analysis of such a local interference of probabilities, one should notice that as a consequence of (157), the equations (102) describing the evolutions of the probabilities f_i, becomes nonlinear, and they can be considered as discretized versions of nonlinear parabolo-hyperbolic equations since the shift operator E is expressed via the differential operator D as:

$$E_h = e^{hD}, \; E_\tau = e^{\tau D} \tag{158}$$

where h and τ are space and time shifts, respectively. That is why the local interference of probabilities can lead to such fundamental nonlinear effects as

shock waves, Burger's waves, solitons, i.e., concentrations of probabilities which can be interpreted as special emerging effects of self-organization.

There are seveal advantages of classical imitations of probability interference over "natural" quantum simulations for non-quantum applications. Firstly, the interference is not restricted to linear unitary (reversible) transformations; moreover, it can be selected in an optimal way subject to the required objective of the performance.

Secondly, it can be applied to a much broader class of problems, and in particular, to the field of intelligent systems which are fundamentally irreversible.

Thirdly, the classical imitations of the probability interference are based upon existing technology: both chaotic and non-Lipschitz dynamics can be implemented in circuits.

So far we discussed non-quantum computational applications. Now one may ask how far can we go in imitating a hypothetical quantum computer by classical devices. The border line for such imitations is drawn by the so called hidden-varriable problem (R. Feynmann, (1982)): It is impossible to represent the results of quantum mechanics with a classical universal device. That means that for applications to quantum mechanics, the superiority of a quantum computer over any classical one is unquestionable.

8 Simulation of Schrödinger equation.

The mathematical formalism exploited in all the previous sections was based upon the relationships between the non-Lipschitz dynamical equations (1), (2) simulating random walk, and the discretized version of the Fokker-Planck equation (12) governing the evolution of the probability corresponding to this random walk.

A continuous version of Eq. (12), i.e., the Fokker-Planck equation, is obtained if

$$\nu \sim \omega^{4/3}, and\ \omega \to \infty \tag{159}$$

Indeed, then:

$$\tau = \frac{2\pi}{\omega} \to 0,\ h \sim \frac{1}{\omega^{1/2}} \to 0,\ \frac{h^2}{\tau} \to 2D = Const \tag{160}$$

and Eq. (12) reduces to the Fokker-Planck equation:

$$\frac{\partial f}{\partial t} = D^2 \frac{\partial^2 f}{\partial x^2} \tag{161}$$

There is a formal mathematical similarity between the Fokker-Planck and the Schrödinger equations: replacing real time t in Eq. (161) by an imaginary time

$$t_{**} = it,\ i = \sqrt{-1} \tag{162}$$

one arrives at the Schrödinger equation:

$$i\frac{\partial f}{\partial t_{**}} = D^2 \frac{\partial^2 f}{\partial x^2} \tag{163}$$

Continuing this analogy, one may ask: does there exist a dynamical system which simulates the Schrödinger equation (1 63) in the same way in which the dynamical system (1),(2) simulates the Fokker-Planck equation?

The formal mathematical answer to this question is very simple: yes, it does. Indeed, turning to Eqs. (1),(2) and introducing an imaginary time

$$t_* = it \tag{164}$$

one obtains

$$\frac{dx}{dt_*} = v_*, \frac{dv_*}{dt} = \nu v_*^{1/3} Sin\omega t_*,\ \nu \sim \omega^{1/3},\ \omega \to \infty \tag{165}$$

Formally this system is identical to Eqs. (1),(2), and therefore, it describes a random walk whose probability is governed by the Fokker-Planck equation:

$$\frac{\partial f}{\partial t_*} = D^2 \frac{\partial^2 f}{\partial x^2} \tag{166}$$

or, after returning to the real time t, by the Schrödinger Equation (163).

Surprisingly, the mechanism of the instability of Eqs. (1 65) is explained much easier here than those in the classical case: it just follows from the uncertainty principle which rejects a possibility that initial conditions for both the position and the velocity of a particle are known exactly.

However, for the purpose of actual simulations of the Schrödinger equation (163), the dynamical system does not offer much (since it evolves in imaginary time), unless it can be given a meaningful physical interpretation. A mathematical formalism for such an interpretation can be borrowed from special theory of relativity in which physical events are mapped into a pseudo-euclidian space with real space coordinates and imaginary time. But the main problem here is not in mathematics, but rather in physics: does the dynamical system (165) exist in real physical world?

The discovery of chaos in classical mechanics raised many questions among quantum physicists about a possibility that there is a deterministic microstructure behind the Schrödinger equation, and as a result of instability, this microstructure loses its determinism and "collapses" into probabilistic world in the same way in which deterministic Newtonian dynamics attains stochasticity due to chaos. Such speculations were encouraged by views expressed by A. Einstein who had never been comfortable with the probabilistic origin of quantum mechanics. From this viewpoint, the dynamical system (165) represents an alternative to this probabilistic origin: it is fully deterministic (since it does not include any random parameters); it is driven by instability triggered by uncertainties in initial conditions (in this context, the uncertainty

principle in quantum mechanics plays the same role as the finite precision of initial conditions does in classical mechanics); and finally, the evolutin of probability resulting from instabilities is described by the Schrödinger equation,

At this stage, we cannot prove (or disprove) existence of a deterministic origin of quantum mechanics. But we can make the following statement: if such a deterministic origin exists, its phenomenological structure is likely to be similar to those of Eqs. (165), and then the quantum device for a quantum computer can be based directly upon special "quantum" simulations of Eqs. (165) as it was described in the previous sections for classical simulations.

"The research described in this paper was performed by the Center for Space Microelectronics Technology, Jet Propulsion Laboratory, California Institute of Technology, under contract with the National Aeronautics and Space Administration, Office of Space Access and Technology."

References

(1) G. Cavello and E. George, The American Statistician, August 1992, Vol 46, No. 3.

(2) R. Feynman, Int. J. of Theoretical Physics, Vol 21, Nos 6/7, 1982

(3) s. Forrest, Physics D 42 (1990), 1-11

(4) H. Haken, Information and Self-Organization, Springer Verlag, 1988

(5) D. Richardson, Journal of Symbolic Logic, 33,514, 1968

(6) P. Shor, in Proceedings of the 35th Annual Symposium on the Foundations of Comp. Science, IEEE Computer Society Press, N.Y. 1994 P. 124

(7) Q. A. Tuchette, C. T. Hood, W. Lange, H. Marbuchi, and H. T, Kimble, Phys. Rev. Lett. Vol. 75, No. 25, 1995

(8) M. Zak, Int. J. of Theoretical Physics, Vol 33, No. 5, 1994

(9) M. Zak, Int. J. of Theoretical Physics, Vol. 35, No 2, 1996

(Io) T. Zbilut, M. Zak, and R. Meyers, Terminal Dynamics Model of the Heartbeat, Biological Cybernetics (in press).

(11) H. Ziegler, Some Extremum Principles in Irreversible Thermodynamics, North-Holland Publ. Co., Amsterdam, 1963